

# ON $\theta$ -CONGRUENT NUMBERS ON REAL QUADRATIC NUMBER FIELDS

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**ABSTRACT.** Let  $\mathbb{K} = \mathbb{Q}(\sqrt{m})$  be a real quadratic number field, where  $m > 1$  is a squarefree integer. Suppose that  $0 < \theta < \pi$  has rational cosine, say  $\cos(\theta) = s/r$  with  $0 < |s| < r$  and  $\gcd(r, s) = 1$ . A positive integer  $n$  is called a  $(\mathbb{K}, \theta)$ -congruent number if there is a triangle, called the  $(\mathbb{K}, \theta, n)$ -triangles, with sides in  $\mathbb{K}$  having  $\theta$  as an angle and  $n\alpha_\theta$  as area, where  $\alpha_\theta = \sqrt{r^2 - s^2}$ . Consider the  $(\mathbb{K}, \theta)$ -congruent number elliptic curve  $E_{n,\theta} : y^2 = x(x + (r + s)n)(x - (r - s)n)$  defined over  $\mathbb{K}$ . Denote the squarefree part of positive integer  $t$  by  $\text{sqf}(t)$ . In this work, it is proved that if  $m \neq \text{sqf}(2r(r - s))$  and  $mn \neq 2, 3, 6$ , then  $n$  is a  $(\mathbb{K}, \theta)$ -congruent number if and only if the Mordell-Weil group  $E_{n,\theta}(\mathbb{K})$  has positive rank, and all of the  $(\mathbb{K}, \theta, n)$ -triangles are classified in four types.

**Keywords:**  $\theta$ -congruent number, elliptic curve, real quadratic number field.

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## 1. INTRODUCTION

A positive integer  $n$  is called a *congruent number* if it is the area of a right triangle with rational sides. Finding all congruent numbers is one of the classical problems in the modern number theory. We cite [8] for an exposition of the congruent number problem, and [4] to see the first study of  $\theta$ -congruent numbers as a generalization of the classic one. Let  $0 < \theta < \pi$  has rational cosine  $\cos(\theta) = s/r$  with  $0 < |s| < r$  and  $\gcd(r, s) = 1$ . Let  $(U, V, W)_\theta$  denote a triangle with an angle  $\theta$  between sides  $U$  and  $V$ . A positive integer  $n$  is called a  *$\theta$ -congruent number* if there exists a triangle  $(U, V, W)_\theta$  with sides in  $\mathbb{Q}$  having area  $n\alpha_\theta$ , where  $\alpha_\theta = \sqrt{r^2 - s^2}$ . In other words,  $n$  is a  $\theta$ -congruent number if it satisfies

$$2rn = UV, \quad W^2 = U^2 + V^2 - \frac{2s}{r}UV.$$

An ordinary congruent number is nothing but a  $\pi/2$ -congruent number. Clearly, if  $n$  is a  $\theta$ -congruent number, then so is  $nt^2$ , for any positive integer  $t$ . We shall concentrate on squarefree numbers whenever  $\theta$ -congruent numbers concerned. Let

$$E_{n,\theta} : y^2 = x(x + (r + s)n)(x - (r - s)n)$$

be the  $\theta$ -congruent number elliptic curve, where  $r$  and  $s$  are as above. Theorem 2.4 gives an important connection between  $\theta$ -congruent numbers and the Mordell-Weil group  $E_{n,\theta}(\mathbb{Q})$ . For more information and recent results about  $\theta$ -congruent numbers see [5, 3, 14].

The notion  $\theta$ -congruent number, which is defined over  $\mathbb{Q}$ , can be extended in a natural way over real quadratic number fields  $\mathbb{K}$ . In this case, we refer to  $n$  as a  $(\mathbb{K}, \theta)$ -congruent number and to the triangle  $(U, V, W)_\theta$  as a  $(\mathbb{K}, \theta, n)$ -triangle. When  $n$  is not a  $\theta$ -congruent number over  $\mathbb{Q}$ , a question proposed naturally: *Is  $n$  a  $(\mathbb{K}, \theta)$ -congruent number for some real quadratic number field  $\mathbb{K}$ ?* Tada [13] answered this question in the case  $\theta = \pi/2$ , by studying the structure of the  $\mathbb{K}$ -rational points on the elliptic curve  $E_{n, \pi/2} : y^2 = x(x^2 - n^2)$ . In this paper, we answer the above question for any  $0 < \theta < \pi$  and classify all  $(\mathbb{K}, \theta, n)$ -triangles. Through the paper we shall consider  $\mathbb{K} = \mathbb{Q}(\sqrt{m})$  to be a real quadratic field, where  $m > 1$  is squarefree. We denote the squarefree part of any positive integer  $N$  by  $\text{sqf}(N)$ . The main results of this paper are the following theorems.

**Theorem 1.1.** *Let  $n$  be a positive squarefree integer with  $\gcd(m, n) = 1$  such that  $mn \neq 2, 3, 6$  and  $m \neq \text{sqf}(2r(r - s))$ , where  $m, r, s$  are as before. Then  $n$  is a  $(\mathbb{K}, \theta)$ -congruent number if and only if  $\text{rank}(E_{n, \theta}(\mathbb{K})) > 0$ . Moreover,  $n$  is a  $(\mathbb{K}, \theta)$ -congruent number if and only if either  $n$  or  $mn$  is a  $\theta$ -congruent number over  $\mathbb{Q}$ .*

Theorem 1.1 is an extension of Part (2) of Theorem 2.4 in the following. Note that the non-equality conditions for  $mn$  and  $m$  in Theorem 1.1 are necessary. For a counterexample, when  $n = 1$  and  $\theta = 2\pi/3$ , we have  $r = 2$ ,  $s = -1$ ,  $\alpha_\theta = \sqrt{3}$ . Now taking  $m = 3 = \text{sqf}(2r(r - s))$ , there is a  $(\mathbb{Q}(\sqrt{3}), \theta, 1)$ -triangle with sides  $(2, 2, 2\sqrt{3})$  and area  $\sqrt{3}$  but using Theorem 2.1,  $\text{rank}(E_{1, \theta}(\mathbb{Q}(\sqrt{3}))) = \text{rank}(E_{1, \theta}(\mathbb{Q})) + \text{rank}(E_{3, \theta}(\mathbb{Q})) = 0$ .

The following theorem classifies all types of  $(\mathbb{K}, \theta, n)$ -triangles.

**Theorem 1.2.** *Assume that  $n$  is not a  $\theta$ -congruent number over  $\mathbb{Q}$  and let  $\sigma$  be the generator of  $\text{Gal}(\mathbb{K}/\mathbb{Q})$ . Then any  $(\mathbb{K}, \theta, n)$ -triangle with  $(U, V, W) \in (\mathbb{K}^*)^3$  and  $(0 < U \leq V < W)$  is necessarily one of the following types:*

Type 1.  $U\sqrt{m}, V\sqrt{m}, W\sqrt{m} \in \mathbb{Q}$ ;

Type 2.  $U, V, W\sqrt{m} \in \mathbb{Q}$ ;

Type 3.  $U, V \in \mathbb{K} \setminus \mathbb{Q}$  such that  $\sigma(U) = V$ ,  $W \in \mathbb{Q}$ ;

Type 4.  $U, V \in \mathbb{K} \setminus \mathbb{Q}$  such that  $\sigma(U) = -V$ ,  $W \in \mathbb{Q}$ .

Let  $A = \text{sqf}(r^2 - s^2)$ ,  $B = \text{sqf}(2r(r - s))$  and  $C = \text{sqf}(2r(r + s))$ . The following proposition shows when there is no  $(\mathbb{K}, \theta, n)$ -triangle of Types 2, 3 and 4.

**Proposition 1.3.** *Let  $p$  be a prime number and the pair  $(m, A)$  (resp.  $(m, B)$  and  $(m, C)$ ) can be written as  $(p^\alpha a, p^\beta b)$ , where  $\alpha, \beta \in \{0, 1\}$  and  $\gcd(p, ab) = 1$ . Then there is no  $(\mathbb{K}, \theta, n)$ -triangle of Type 2 (resp. Type 3 and Type 4) whenever one of the following conditions hold.*

(1)  $p = 2$  :  $(\alpha, \beta) = (0, 0)$  and  $(a, b) \stackrel{4}{\equiv} (3, 3)$ ,

$(\alpha, \beta) = (0, 1)$  and  $(a, b) \stackrel{8}{\equiv} (3, 1), (3, 5), (7, 5), (7, 7)$ ,

$(\alpha, \beta) = (1, 0)$  and  $(a, b) \stackrel{8}{\equiv} (1, 3), (1, 5), (3, 5), (3, 7), (5, 3), (5, 7), (7, 3), (7, 7)$ ,

$(\alpha, \beta) = (1, 1)$  and  $(a, b) \stackrel{8}{\equiv} (1, 3), (1, 5), (3, 1), (3, 3), (5, 1), (5, 7), (7, 5), (7, 7)$ ;

- (2)  $p \equiv 1 \pmod{4} : (\alpha, \beta) = (0, 1)$  and  $\left(\frac{a}{p}\right) = -1$ ,  $(\alpha, \beta) = (1, 0)$  and  $\left(\frac{b}{p}\right) = -1$ ,  
 $(\alpha, \beta) = (1, 1)$  and  $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = -1$ ;
- (3)  $p \equiv 3 \pmod{4} : (\alpha, \beta) = (0, 1)$  and  $\left(\frac{a}{p}\right) = -1$ ,  $(\alpha, \beta) = (1, 0)$  and  $\left(\frac{b}{p}\right) = -1$ ,  
 $(\alpha, \beta) = (1, 1)$  and  $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = 1$ .

The next result settles a condition on  $n$  and  $mn$  to be  $\theta$ -congruent over  $\mathbb{Q}$ .

**Theorem 1.4.** *Let  $n$  be a positive squarefree integer such that  $\gcd(m, n) = 1$  and  $mn \neq 2, 3, 6$ . Then the following statements are equivalent.*

- (1) *There is a  $(\mathbb{K}, \theta, n)$ -triangle  $(U, V, W)_\theta$  with  $0 < U \leq V < W$ ,  $W \notin \mathbb{Q}$  and  $W\sqrt{m} \notin \mathbb{Q}$ ;*
- (2) *The integers  $n$  and  $mn$  are  $\theta$ -congruent numbers over  $\mathbb{Q}$ .*

## 2. PRELIMINARIES

Consider an elliptic curve  $E : y^2 = x^3 + ax^2 + bx + c$  over  $\mathbb{Q}$ . Recall that the  $m$ -twist  $E^m$  of  $E$  is an elliptic curve over  $\mathbb{Q}$  defined by  $y^2 = x^3 + amx^2 + bm^2x + cm^3$ . The next result establishes a fact about ranks [10].

**Theorem 2.1.** *Let  $E$  be an elliptic curve over  $\mathbb{Q}$ . Then*

$$\text{rank}(E(\mathbb{K})) = \text{rank}(E(\mathbb{Q})) + \text{rank}(E^m(\mathbb{Q})).$$

We denote the torsion subgroup of the groups  $E(\mathbb{K})$  and  $E^m(\mathbb{K})$  by  $T(E, \mathbb{K})$  and  $T(E^m, \mathbb{K})$ , respectively. Also, we write  $T_{n,\theta}(\mathbb{K})$  and  $T_{n,\theta}^m(\mathbb{K})$ , respectively, in the case  $E = E_{n,\theta}$ . The following proposition and theorem have essential roles in the proof of our results.

**Proposition 2.2** ([9, Proposition 1]). *Let  $E$  be an elliptic curve over  $\mathbb{K}$ . Then the map*

$$\phi : T(E, \mathbb{K})/T(E, \mathbb{Q}) \rightarrow T(E^m, \mathbb{Q}), \quad \phi(\tilde{P}) := P - \sigma(P)$$

*is an injective map of abelian groups, where  $\sigma$  is the generator of  $\text{Gal}(\mathbb{K}/\mathbb{Q})$ .*

**Theorem 2.3** ([7, Theorem 4.2]). *Let  $\mathbb{F}$  be an algebraic number field and  $E$  an elliptic curve over  $\mathbb{F}$  defined by*

$$y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3), \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}.$$

*Suppose that  $(x_0, y_0)$  be an  $\mathbb{F}$ -rational point of  $E$ . Then, there exists an  $\mathbb{F}$ -rational point  $(x_1, y_1)$  with  $2(x_1, y_1) = (x_0, y_0)$  if and only if  $x_0 - \alpha_1, x_0 - \alpha_2, x_0 - \alpha_3$  are squares in  $\mathbb{F}$ .*

The next results give important information about  $\theta$ -congruent numbers over  $\mathbb{Q}$ .

**Theorem 2.4.** (Fujiwara, [4]) *Consider  $0 < \theta < \pi$  with rational cosine.*

- (1) *A positive integer  $n$  is a  $\theta$ -congruent number if and only if  $E_{n,\theta}(\mathbb{Q})$  has a point of order greater than 2;*
- (2) *If  $n \neq 1, 2, 3, 6$ , then  $n$  is a  $\theta$ -congruent number if and only if  $E_{n,\theta}(\mathbb{Q})$  has positive rank.*

All possibilities for the torsion subgroup of  $E_{n,\theta}(\mathbb{Q})$  can be found in the next result.

**Theorem 2.5.** (Fujiwara, [5]) *Let  $T_{n,\theta}(\mathbb{Q})$  be the torsion subgroup of the  $\theta$ -congruent number elliptic curve  $E_{n,\theta}$  over  $\mathbb{Q}$ .*

- (1)  $T_{n,\theta}(\mathbb{Q}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_8$  if and only if there exist integers  $a, b > 0$  such that  $\gcd(a, b) = 1$ ,  $a$  and  $b$  have opposite parity and satisfy either of the following conditions.
  - (i)  $n = 1$ ,  $r = 8a^4b^4$ ,  $r - s = (a - b)^4$ ,  $(1 + \sqrt{2})b > a > b$ ,
  - (ii)  $n = 2$ ,  $r = (a^2 - b^2)^4$ ,  $r - s = 32a^4b^4$ ,  $a > (1 + \sqrt{2})b$ ;
- (2)  $T_{n,\theta}(\mathbb{Q}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6$  if and only if there exist integers  $u, v > 0$  such that  $\gcd(u, v) = 1$ ,  $u > 2v$  and satisfy one of the following conditions:
  - (i)  $n = 1$ ,  $r = \frac{1}{2}(u - v)^3(u + v)$ ,  $r + s = u^3(u - 2v)$ ,
  - (ii)  $n = 2$ ,  $r = (u - v)^3(u + v)$ ,  $r + s = 2u^3(u - 2v)$ ,
  - (iii)  $n = 3$ ,  $r = \frac{1}{6}(u - v)^3(u + v)$ ,  $r + s = \frac{1}{3}u^3(u - 2v)$ ,
  - (iv)  $n = 6$ ,  $r = \frac{1}{3}(u - v)^3(u + v)$ ,  $r + s = \frac{2}{3}u^3(u - 2v)$ ;
- (3)  $T_{n,\theta}(\mathbb{Q}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$  if and only if either of the following holds.
  - (i)  $n = 1$ ,  $2r$  and  $r - s$  are squares but not satisfy (i) of Part (1),
  - (ii)  $n = 2$ ,  $r$  and  $2(r - s)$  are squares but not satisfy (ii) of Part (1);
- (4) Otherwise,  $T_{n,\theta}(\mathbb{Q}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ .

**Remark 2.6.** For any squarefree integer  $m > 1$ , the  $m$ -twist  $E_{n,\theta}^m$  of the elliptic curve  $E_{n,\theta}$  is defined by  $y^2 = x(x + (r + s)mn)(x - (r - s)mn)$  which is equal to  $E_{mn,\theta}$ , as seen. Therefore  $E_{n,\theta}^m(\mathbb{Q}) = E_{mn,\theta}(\mathbb{Q})$ , and hence  $T_{n,\theta}^m(\mathbb{Q}) = T_{mn,\theta}(\mathbb{Q})$ .

### 3. PROOFS

Appealing to Proposition 2.2, we first settle all possibilities for the torsion subgroup of  $E_{n,\theta}(\mathbb{K})$ . Let  $h, k$ , and  $d$  be integers such that  $2r = h^2 \text{sqf}(2r)$ ,  $r - s = k^2 \text{sqf}(r - s)$  and  $2r(r - s) = d^2 m$ , where  $m = \text{sqf}(2r(r - s))$ .

**Proposition 3.1.** Assume that  $m > 1$  and  $n$  are squarefree positive integers such that  $\gcd(m, n) = 1$  and  $mn \neq 2, 3, 6$ . Let  $T_{n,\theta}(\mathbb{K})$  be the torsion subgroup of  $E_{n,\theta}(\mathbb{K})$ .

- (1) If  $m = \text{sqf}(2r(r - s))$  and  $n = \text{sqf}(2r)$ , then

$$T_{n,\theta}(\mathbb{K}) = \{\infty, (0, 0), (-(r + s)n, 0), ((r - s)n, 0), \\ ((nh)^2 - nd\sqrt{m}, \pm(\frac{d^2mn}{h} - n^2hd\sqrt{m})), ((nh)^2 + nd\sqrt{m}, \pm(\frac{d^2mn}{h} + n^2hd\sqrt{m}))\};$$

- (2) If  $m = \text{sqf}(2r(r - s))$  and  $n = \text{sqf}(r - s)$ , then

$$T_{n,\theta}(\mathbb{K}) = \{\infty, (0, 0), (-(r + s)n, 0), ((r - s)n, 0), \\ ((nk)^2 - nd\sqrt{m}, \pm(\frac{d^2mn}{k} - n^2kd\sqrt{m})), ((nk)^2 + nd\sqrt{m}, \pm(\frac{d^2mn}{k} + n^2kd\sqrt{m}))\};$$

- (3) Otherwise,  $T_{n,\theta}(\mathbb{K}) = \{\infty, (0, 0), (-(r + s)n, 0), ((r - s)n, 0)\}$ .

*Proof.* The 2-torsion subgroup of  $E_{n,\theta}(\mathbb{K})$  is:

$$E_{n,\theta}[2](\mathbb{K}) = \{\infty, (0, 0), (-(r + s)n, 0), ((r - s)n, 0)\}.$$

Therefore, we have  $T_{n,\theta}(\mathbb{K}) \supset E_{n,\theta}[2](\mathbb{K}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ . By Remark 2.6 and Theorem 2.5,  $T_{n,\theta}^m(\mathbb{Q}) = T_{mn,\theta}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ . Since  $T_{n,\theta}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ , by Proposition 2.2 and [9, Theorem 1] we have

$$T_{n,\theta}(\mathbb{K}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \text{ or } \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

First let  $T_{n,\theta}(\mathbb{K}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ . Then there exists a point  $P = (x_0, y_0)$  of order 4 in  $T_{n,\theta}(\mathbb{K})$ . Then  $2P$  must be one of the points  $(0, 0)$ ,  $-(r+s)n, 0$  and  $((r-s)n, 0)$ . If  $2P = (0, 0)$  then both  $(r+s)n$  and  $-(r-s)n$  are squares in  $\mathbb{K}$ , which is impossible since  $\mathbb{K}$  is a real quadratic number field and hence  $-1$  is not a square in  $\mathbb{K}$ . Similarly, if  $2P = (-(r+s)n, 0)$ , then  $-(r+s)n$  and  $-2rn$  are squares in  $\mathbb{K}$ , again a contradiction by the same reason. If  $2P = ((r-s)n, 0)$ , then  $(r-s)n$  and  $2rn$  are squares in  $\mathbb{K}$ . Since  $n$  is squarefree, these integers are squares in  $\mathbb{K}$  if  $m = \text{sqf}(2r(r-s))$ . By a simple computation using the duplication formula we obtain (1) and (2). Now,  $T_{n,\theta}(\mathbb{K}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  implies (3), and the proof is completed.  $\square$

**Proof of Theorem 1.1.** Consider the two sets

$$S = \{(U, V, W) \in (\mathbb{K}^*)^3 : 0 < U \leq V < W, UV = 2rn \text{ and } U^2 + V^2 - 2sUV/r = W^2\},$$

$$T = \{(u, v) \in 2E_{n,\theta}(\mathbb{K}) \setminus \{\infty\} : v \geq 0\}.$$

There is a one to one correspondence between the two sets  $S$  and  $T$  via the two mutually inverse maps  $\varphi : S \rightarrow T$  and  $\psi : T \rightarrow S$  defined by

$$\varphi(U, V, W) := (W^2/4, W(V^2 - U^2)/8),$$

$$\psi(u, v) := (\sqrt{u + (r+s)n} - \sqrt{u - (r-s)n}, \sqrt{u + (r+s)n} + \sqrt{u - (r-s)n}, 2\sqrt{u}).$$

Clearly,  $E_{n,\theta}(\mathbb{K}) \setminus E_{n,\theta}[2](\mathbb{K}) \neq \emptyset$  if and only if  $S \neq \emptyset$ .

Suppose that  $m \neq \text{sqf}(2r(r-s))$  and  $mn \neq 2, 3, 6$ . Then by proposition 3.1, we have  $T_{n,\theta}(\mathbb{K}) = E_{n,\theta}[2](\mathbb{K})$ . Therefore,  $\text{rank}(E_{n,\theta}(\mathbb{K})) > 0$  if and only if  $E_{n,\theta}(\mathbb{K}) \setminus E_{n,\theta}[2](\mathbb{K}) \neq \emptyset$ . So  $\text{rank}(E_{n,\theta}(\mathbb{K})) > 0$  if and only if either  $\text{rank}(E_{n,\theta}(\mathbb{Q})) > 0$  or  $\text{rank}(E_{n,\theta}^m(\mathbb{Q})) > 0$ , by Theorem 2.1. The second part of the theorem follows from Remark 2.6.  $\square$

**Proof of Theorem 1.2.** Assume  $n$  is a  $(\mathbb{K}, \theta)$ -congruent number and  $(U, V, W)_\theta$  is the corresponding  $(\mathbb{K}, \theta, n)$ -triangle with area  $n\alpha_\theta$  such that  $0 < U \leq V < W$ . As in the proof of the Theorem 1.1, there is a point  $P = (x, y)$  in  $E_{n,\theta}(\mathbb{K}) \setminus E_{n,\theta}[2](\mathbb{K})$  such that  $\psi(P) = (U, V, W)$ . Substituting  $P$  by  $P + (0, 0)$ ,  $P + (-(r+s)n, 0)$  or  $P + ((r-s)n, 0)$ , if necessary, we may assume that  $x > [(r+s) + \sqrt{2r(r-s)}]$ . Putting  $2P = (u, v)$  and using the map  $\psi$  in the proof of Theorem 1.1, we obtain

$$U = 2rnx/|y|, \quad V = x^2 + 2snx - (r^2 - s^2)n^2/|y|, \quad W = x^2 + (r^2 - s^2)n^2/|y|,$$

where  $x, y \in \mathbb{K}$  and  $|\cdot|$  is the usual absolute value induced from the embedding  $\iota : \mathbb{K} \hookrightarrow \mathbb{R}$  with  $\iota(\sqrt{m})$  positive. Suppose  $\sigma$  is a generator of  $\text{Gal}(\mathbb{K}/\mathbb{Q})$  and put  $\sigma(P) = (\sigma(x), \sigma(y))$ . Since  $P + \sigma(P)$  is an element of  $E_{n,\theta}(\mathbb{Q})$  and  $n$  is not a  $\theta$ -congruent number,  $P + \sigma(P) \in T_{n,\theta}(\mathbb{Q}) = \{\infty, (0, 0), (-(r+s)n, 0), ((r-s)n, 0)\}$ . Hence, one of the following cases necessarily happens:

- I.  $P + \sigma(P) = \infty$ . In this case,  $\sigma(x) = x$  and  $\sigma(y) = -y$ . So,  $x, y\sqrt{m}$  and hence  $U\sqrt{m}, V\sqrt{m}$  and  $W\sqrt{m}$  are rational and we obtain a  $(\mathbb{K}, \theta, n)$ -triangle of Type 1.
- II.  $P + \sigma(P) = (0, 0)$ . We have  $\sigma(x)/x = \sigma(y)/y$ , which we denote by  $\alpha$ . Then,

$$\sigma(y)^2 = \alpha^2 y^2 = \alpha^2 x^3 + 2sn\alpha^2 x^2 - (r^2 - s^2)n^2 \alpha^2 x.$$

Since  $\sigma(P)$  is a point on  $E_{n,\theta}$ , we get

$$\sigma(y)^2 = \sigma(x)^3 + 2sn\sigma(x)^2 - (r^2 - s^2)n^2 \sigma(x) = \alpha^3 x^3 + 2sn\alpha^2 x^2 - (r^2 - s^2)n^2 \alpha x.$$

Clearly,  $\alpha \neq 0, 1$  and  $x \neq 0$ , which implies  $x\sigma(x) = \alpha x^2 = -(r^2 - s^2)n^2$ . Therefore,

$$V = x(x + 2sn + \sigma(x))/|y|, \quad W\sqrt{m} = x(x - \sigma(x))\sqrt{m}/|y|.$$

Since  $x/y = \sigma(x/y)$  and  $x > [(r + s) + \sqrt{2r(r - s)}]n$ , then  $x/|y|$  is rational and hence  $U = 2rn x/|y|$ ,  $V$  and  $W\sqrt{m}$  are rational, which gives a  $(\mathbb{K}, \theta, n)$ -triangle of Type 2.

- III.  $P + \sigma(P) = ((r - s)n, 0)$ . We have  $\sigma(x - (r - s)n)/(x - (r - s)n) = \sigma(y)/y$ , which we denote by  $\beta$ . Put  $z = x - (r - s)n$ . Then,

$$\sigma(y)^2 = \beta^2 [z^3 + (3r - s)n z^2 + 2r(r - s)n^2 z].$$

Since  $\sigma(P)$  is a point on  $E_{n,\theta}$ , we get

$$\sigma(y)^2 = \beta^3 z^3 + (3r - s)n\beta^2 z^2 + 2r(r - s)n^2 \beta z.$$

Now  $\beta \neq 0, 1$  and  $z \neq 0$ , which implies  $\beta z^2 = 2r(r - s)n^2$ . Substituting this equation and  $x = z + (r - s)n$  in  $U$ ,  $V$  and  $W$ , we obtain

$$U = \frac{z(\sigma(z) + 2rn)}{|y|}, \quad V = \frac{z(z + 2rn)}{|y|}, \quad W = \frac{z(z + 2(r - s)n + \sigma(z))}{|y|}.$$

Since  $z/y = \sigma(z/y)$  and  $z > 0$ , then  $z/|y|$  and hence  $W$  is rational and  $\sigma(U) = V$ . This time we obtain a  $(\mathbb{K}, \theta, n)$ -triangle of Type 3.

- IV.  $P + \sigma(P) = (-(r + s)n, 0)$ . Put  $w = x + (r + s)n$ . As in Case III,  $w/|y|$  and

$$W = w(w - 2(r + s)n + \sigma(w))/|y|$$

are rational and  $\sigma(U) = -V$ , where

$$U = w(2rn - \sigma(w))/|y|, \quad V = w(w - 2rn)/|y|.$$

Therefore, we obtain a  $(\mathbb{K}, \theta, n)$ -triangle of Type 4. □

**Proof of Proposition 1.3.** If we suppose that there is a  $(\mathbb{K}, \theta, n)$ -triangle of Type 2, say  $(U, V, W)_\theta = (u, v, w\sqrt{m})$  with  $u, v, w \in \mathbb{Q}^+$ , then  $(x, y, z) = (ru - sv, v, mrw)$  is a non-zero solution of the equation

$$z^2 = mx^2 + m(r^2 - s^2)y^2. \tag{3.1}$$

And, if there is a  $(\mathbb{K}, \theta, n)$ -triangle of Type 3, say  $(U, V, W)_\theta = (u - v\sqrt{m}, u + v\sqrt{m}, w)$  such that  $\sigma(U) = V$ , then  $(x, y, z) = (u, v, rw)$  is a non-zero solution of

$$z^2 = 2r(r - s)x^2 + 2mr(r + s)y^2. \quad (3.2)$$

Similarly, if  $(U, V, W)_\theta = (-u + v\sqrt{m}, u + v\sqrt{m}, w)$  is a  $(\mathbb{K}, \theta, n)$ -triangle of Type 4 such that  $\sigma(U) = -V$ , then  $(x, y, z) = (u, v, rw)$  satisfies

$$z^2 = 2r(r + s)x^2 + 2mr(r - s)y^2. \quad (3.3)$$

By the Hasse local-global principle, the equations (3.1), (3.2) and (3.3) have solutions in  $\mathbb{Q}$  if and only if they have a solution in  $\mathbb{Q}_p$  for every prime  $p$ , where  $\mathbb{Q}_p$  is the field of  $p$ -adic numbers. We assume that  $A = \text{sqf}(r^2 - s^2)$ , and for a prime  $p$  the pair  $(m, A)$  ( $(m, B)$ , and  $(m, C)$ , resp.) can be written as  $(p^\alpha a, p^\beta b)$ , where  $\alpha, \beta \in \{0, 1\}$  and  $\gcd(p, a, b) = 1$ . Then, using Hilbert symbols [11, Theorem 1, III], the equations (3.1), (3.2) and (3.3) have solutions in  $\mathbb{Q}_2$  if and only if one of the following cases happens:

- i)  $(\alpha, \beta) = (0, 0)$  and  $(a, b) \not\equiv_8 (3, 3)$ ;
- ii)  $(\alpha, \beta) = (0, 1)$  and  $(a, b) \not\equiv_8 (3, 1), (3, 5), (7, 5), (7, 7)$ ;
- iii)  $(\alpha, \beta) = (1, 0)$  and  $(a, b) \not\equiv_8 (1, 3), (1, 5), (3, 5), (3, 7), (5, 3), (5, 7), (7, 3), (7, 7)$ ;
- iv)  $(\alpha, \beta) = (1, 1)$  and  $(a, b) \not\equiv_8 (1, 3), (1, 5), (3, 1), (3, 3), (5, 1), (5, 7), (7, 5), (7, 7)$ .

Also, the equations (3.1), (3.2) and (3.3) have solutions in  $\mathbb{Q}_p$  with  $p \equiv 1 \pmod{4}$  if and only if one of the following happens:

- i)  $(\alpha, \beta) = (0, 1)$  and  $\left(\frac{a}{p}\right) = 1$ ;
- ii)  $(\alpha, \beta) = (1, 0)$  and  $\left(\frac{b}{p}\right) = 1$ ;
- iii)  $(\alpha, \beta) = (1, 1)$  and  $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = 1$ .

□

**Proof of Theorem 1.4. Case 1.**  $n$  and  $mn$  are  $(\mathbb{Q}, \theta)$ -congruent numbers. Consider the  $(\mathbb{Q}, \theta, n)$ -triangle  $(U_1, V_1, W_1)_\theta$  and the  $(\mathbb{Q}, \theta, mn)$ -triangle  $(U_2, V_2, W_2)_\theta$ , where

$$0 < U_1 \leq V_1 < W_1, \quad 2rn = U_1V_1, \quad U_1^2 + V_1^2 - \frac{2sU_1V_1}{r} = W_1^2,$$

$$0 < U_2 \leq V_2 < W_2, \quad 2rmn = U_2V_2, \quad U_2^2 + V_2^2 - \frac{2sU_2V_2}{r} = W_2^2.$$

Hence,  $(U_2/\sqrt{m}, V_2/\sqrt{m}, W_2/\sqrt{m})_\theta$  is a  $(\mathbb{K}, \theta, n)$ -triangle. Recall the maps  $\varphi$  and  $\psi$  in the proof of Theorem 1.2 and put

$$P = (u, v) = \varphi((U_1, V_1, W_1)) + \varphi((U_2/\sqrt{m}, V_2/\sqrt{m}, W_2/\sqrt{m})).$$

Then the additive law on  $E_{n, \theta}(\mathbb{K})$  implies  $u = a + b\sqrt{m}$ , where

$$a = \frac{m^3W_1^2(V_1^2 - U_1^2)^2 + W_2^2(V_2^2 - U_2^2)^2}{4m(W_2^2 - mW_1^2)^2} - \left(\frac{W_1^2}{4} + \frac{W_2^2}{4m} + 2sn\right) > 0,$$

$$b = -\frac{W_1 W_2 (V_1^2 - U_1^2)(V_2^2 - U_2^2)\sqrt{m}}{2(W_2^2 - mW_1^2)^2}.$$

We may assume  $v \geq 0$ . Since  $(u, v) \in T$ , then  $\psi((u, v)) \in S$  which indicates the sides of a  $(\mathbb{K}, \theta, n)$ -triangle  $(U, V, W)_\theta$ . In fact, if we suppose  $U = u_1 + u_2\sqrt{m}$ ,  $V = v_1 + v_2\sqrt{m}$  and  $W = w_1 + w_2\sqrt{m}$ , where  $u_1, u_2, v_1, v_2, w_1, w_2$  are rational, then

$$w_1 = \pm \sqrt{2(a \pm \sqrt{a^2 - mb^2})}, \quad w_2 = \frac{2b}{w_1},$$

and

$$U = (\alpha_1 - \beta_1) + (\alpha_2 - \beta_2)\sqrt{m}, \quad V = (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2)\sqrt{m},$$

where

$$\alpha_1 = \pm \sqrt{\frac{(a + (r + s)n) \pm \sqrt{(a + (r + s)n)^2 - mb^2}}{2}}, \quad \alpha_2 = \frac{b}{2\alpha_1},$$

$$\beta_1 = \pm \sqrt{\frac{(a - (r - s)n) \pm \sqrt{(a - (r - s)n)^2 - mb^2}}{2}}, \quad \beta_2 = \frac{b}{2\beta_1}.$$

Conversely, suppose to the contrary that  $n$  or  $mn$  is not  $\theta$ -congruent over  $\mathbb{Q}$ . First, assume  $n$  is not  $\theta$ -congruent over  $\mathbb{Q}$  but  $mn$  is  $\theta$ -congruent over  $\mathbb{Q}$ . By Theorem 1.2 (1), there is no  $(\mathbb{K}, \theta, n)$ -triangle  $(U, V, W)_\theta$  satisfying the conditions  $0 < U \leq V < W$ ,  $W \notin \mathbb{Q}$  and  $W\sqrt{m} \notin \mathbb{Q}$ .

**Case 2.**  $mn$  is not  $\theta$ -congruent over  $\mathbb{Q}$  but  $n$  is  $(\mathbb{K}, \theta)$ -congruent. Let  $(U, V, W)_\theta$  denotes the sides of the corresponding  $(\mathbb{K}, \theta, n)$ -triangle. Multiplying the three sides by  $\sqrt{m}$ , we get the  $(\mathbb{K}, \theta, mn)$ -triangle  $(U\sqrt{m}, V\sqrt{m}, W\sqrt{m})_\theta$ . For the positive integer  $mn$ , we define the map  $\varphi'$  in the same way as  $\varphi$ . Put

$$2P' = \varphi'((U\sqrt{m}, V\sqrt{m}, W\sqrt{m}))$$

for some point  $P' \in E_{mn, \theta}(\mathbb{K})$ . For the generator  $\sigma$  of  $\text{Gal}(\mathbb{K}/\mathbb{Q})$ , since  $P' + \sigma(P')$  is an element in  $E_{mn, \theta}(\mathbb{Q})$  and  $mn$  is not  $\theta$ -congruent over  $\mathbb{Q}$ , we have

$$P' + \sigma(P') \in T_{mn, \theta}(\mathbb{Q}) = \{\infty, (0, 0), (-(r + s)mn, 0), ((r - s)mn, 0)\}.$$

Therefore, by the same way as in the proof of Theorem 1.2, one of the following cases necessarily happens:

Type 1.  $U, V, W \in \mathbb{Q}$ ;

Type 2.  $U\sqrt{m}, V\sqrt{m}, W \in \mathbb{Q}$ ;

Type 3.  $U, V \in K \setminus \mathbb{Q}$  such that  $\sigma(U) = V, W\sqrt{m} \in \mathbb{Q}$ ;

Type 4.  $U, V \in K \setminus \mathbb{Q}$  such that  $\sigma(U) = -V, W\sqrt{m} \in \mathbb{Q}$ .

Hence, there is no  $(\mathbb{K}, \theta, n)$ -triangle  $(U, V, W)_\theta$  with  $W \notin \mathbb{Q}$  and  $W\sqrt{m} \notin \mathbb{Q}$ .

**Case 3.** Both  $n$  and  $mn$  are not  $\theta$ -congruent numbers over  $\mathbb{Q}$ , where  $mn \neq 2, 3, 6$ . If  $m \neq \text{sqf}(2r(r - s))$ , by Theorem 1.1,  $n$  is not  $(\mathbb{K}, \theta, n)$ -congruent. If  $m = \text{sqf}(2r(r - s))$  and  $n$  is  $(\mathbb{K}, \theta, n)$ -congruent, we have  $U = V$  for all  $(\mathbb{K}, \theta, n)$ -triangles  $(U, V, W)_\theta$ . Hence, there is no any  $(\mathbb{K}, \theta, n)$ -triangle  $(U, V, W)_\theta$  with  $W \notin \mathbb{Q}$  and  $W\sqrt{m} \notin \mathbb{Q}$ . We have completed the proof of Theorem 1.4.  $\square$



## 4. EXAMPLES

In this section, we give some examples of  $(\mathbb{K}, \theta)$ -congruent numbers and verify all four types of  $(\mathbb{K}, \theta, n)$ -triangles in Theorem 1.2 in the cases  $\theta = \pi/3, 2\pi/3$ . Given  $n$ , let  $(U, V, W)_\theta$  be a  $(\mathbb{K}, \theta, n)$ -triangle. Then, we have

$$0 < U \leq V < W, \quad UV = 2rn, \quad W^2 = U^2 + V^2 - \frac{2s}{r}UV.$$

For any  $(U, V, W)_\theta$ ,  $\varphi((U, V, W)) = (W^2/4, W(V^2 - U^2)/8)$  is a point of  $2E_{n,\theta}(\mathbb{K}) \setminus \{\infty\}$ . Also, for any point  $(u, v) \in 2E_{n,\theta}(\mathbb{K}) \setminus \{\infty\}$ ,

$$\psi((u, v)) = ((\sqrt{u + (r+s)n} - \sqrt{u - (r-s)n}, \sqrt{u + (r+s)n} + \sqrt{u - (r-s)n}, 2\sqrt{u})).$$

In our computations we have used Cremona's MWrnk program [2] and the number theoretic Pari software [1].

**I) Case  $\theta = \pi/3$ .** In this case, we have  $r = 2$ ,  $s = 1$ , and  $\alpha_\theta = \sqrt{3}$ , and hence the area of any  $(\mathbb{K}, \pi/3, n)$ -triangle is  $n\sqrt{3}$ .

**Example 4.1.** Take  $n = 3$  and  $m = 13$ . We have the following  $(\mathbb{Q}(\sqrt{13}), \pi/3, 3)$ -triangles of types 1, 2, 3 and 4 in Theorem 1.1 and the corresponding points in the set  $2E_{3,\pi/3}(\mathbb{Q}(\sqrt{13})) \setminus \{\infty\}$ .

Type 1. An easy computing shows that the rank of  $E_{39,\pi/3}(\mathbb{Q})$  is 2, and the generators of the group are  $P_1 = [-9, -216]$  and  $P_2 = [75, -720]$ . We have

$$2P_1 = [1894/16, -91805/64] \in 2E_{39,\theta}(\mathbb{Q}) \setminus \{\infty\}.$$

Now, using the map  $\varphi$  and  $\psi$ , defined in the proof of the Theorem 1.1 we get a rational  $\pi/3$ -triangle  $(13/2, 24, 43/2)$  with area 39, which gives the following  $(\mathbb{Q}(\sqrt{13}), \pi/3, 3)$ -triangle of Type 1:

$$(U, V, W)_{\pi/3} = (\sqrt{13}/2, 24\sqrt{13}/13, 43\sqrt{13}/26)$$

which corresponds to the following point  $Q = (1894/208, 91805\sqrt{13}/416)$ .

Type 2. We have a  $(\mathbb{Q}(\sqrt{13}), \pi/3, 3)$ -triangle  $(U, V, W)_{\pi/3} = (3, 4, \sqrt{13})$  of type 2 with the corresponding point  $Q = (13/4, 7\sqrt{13}/8)$ .

Type 3. Let  $U = u - v\sqrt{13}$ ,  $V = u + v\sqrt{13}$  and  $W = w$ , where  $u, v, w \in \mathbb{Q} \setminus \{0\}$ . Then the pair  $(u, v)$  satisfies the equation  $u^2 - 13v^2 = 12$ . An easy solution of this equation is  $(u_0, v_0) = (5, 1)$ . Parametrizing  $u$  and  $v$  in terms of  $t \in \mathbb{Q}$  we obtain  $u = -5t^2 + 26t - 65/t^2 - 13$  and  $v = t^2 - 10t + 13/t^2 - 13$ . By putting these into  $w^2 = u^2 + 39v^2$  and taking  $t = 13/4$  one can see that  $w^2 = u^2 + 39v^2$  is a square in  $\mathbb{Q}$ . So, we obtain  $(U, V, W)_{\pi/3} = (41 - 11\sqrt{13}/3, 41 + 11\sqrt{13}/3, 80/3)$ , with  $(\mathbb{Q}(\sqrt{13}), \pi/3, 3)$ -triangle of type 3 with the corresponding point  $Q = (1600/3, 18040\sqrt{13}/9)$ .

Type 4. Let  $U = -u + v\sqrt{13}$ ,  $V = u + v\sqrt{13}$  and  $W = w$ , where  $u, v, w \in \mathbb{Q} \setminus \{0\}$ . Then the pair  $(u, v)$  satisfies  $13v^2 - u^2 = 12$  with a solution  $(u_0, v_0) = (1, 1)$ . A similar discussion as in the previous step, taking  $t = 8$ , leads

us to a  $(\mathbb{Q}(\sqrt{13}), \pi/3, 3)$ -triangle of Type 4, with the corresponding point  $Q = (24964/51, 1002352\sqrt{13}/51)$ .

**Example 4.2.** Let  $n = 11$  and  $m = 5$ . One can see that  $n$  is  $\pi/3$ -congruent over  $\mathbb{Q}$  and there is a  $(\mathbb{Q}, \pi/3, 11)$ -triangle  $(U_1, V_1, W_1) = (55/12, 48/5, 499/60)$ . Also,  $nm = 55$  is  $\pi/3$ -congruent over  $\mathbb{Q}$  and  $(U_2, V_2, W_2) = (8, 55/2, 49/2)$  is a rational  $\pi/3$ -triangle with area  $11\sqrt{3}$ . Dividing its sides by  $\sqrt{5}$ , we obtain a  $(\mathbb{Q}(\sqrt{5}), \pi/3, 11)$ -triangle

$$(U_2/\sqrt{5}, V_2/\sqrt{5}, W_2/\sqrt{5}) = (8\sqrt{5}/5, 11\sqrt{5}/2, 49\sqrt{5}/10).$$

Now, a calculations as in the proof of Theorem 1.4 leads to a  $(\mathbb{Q}(\sqrt{5}), \pi/3, 11)$ -triangle

$$(U, V, W) = \left( \frac{1}{310}(1470+499\sqrt{5}), \frac{88}{5909}(1470-499\sqrt{5}), \frac{1}{183179}(4145193-12554399\sqrt{5}) \right)$$

satisfying in Theorem 1.4.

**II) Case  $\theta = 2\pi/3$ .** In this case, we have  $r = 2$ ,  $s = -1$ , and  $\alpha_\theta = \sqrt{3}$ . So, as in the case I, the area of any  $(\mathbb{K}, 2\pi/3, n)$ -triangle is  $n\sqrt{3}$ .

**Example 4.3.** Take  $n = 17$  and  $m = 13$ . By a similar way as in Example 4.1, we find the following  $(\mathbb{Q}(\sqrt{13}), 2\pi/3, 17)$ -triangles with area  $17\sqrt{13}$  of types 1, 2, 3 and 4 preceding by their corresponding points in  $2E_{17, 2\pi/3}(\mathbb{Q}(\sqrt{13})) \setminus \{\infty\}$ .

Type 1.  $(U, V, W)_{2\pi/3} = (17\sqrt{13}/26, 8\sqrt{13}, 217\sqrt{13}/26),$

$$Q = (47089/16, 9325575\sqrt{13}/10816);$$

Type 2.  $(U, V, W)_{2\pi/3} = (1, 68, 19\sqrt{13}), Q = (13/4, 7\sqrt{13}/8);$

Type 3.  $(U, V, W)_{2\pi/3} = (9 - \sqrt{13}, 9 + \sqrt{13}, 16), Q = (64, 72\sqrt{13});$

Type 4.  $(U, V, W)_{2\pi/3} = (-5 + 7\sqrt{13}/3, 5 + 7\sqrt{13}/3, 44/3), Q = (484/9, 770\sqrt{13}/27).$

**Example 4.4.** Let  $n = 19$  and  $m = 6$ . Then 19 is a  $2\pi/3$ -congruent number over  $\mathbb{Q}$  and there is a  $(\mathbb{Q}, 2\pi/3, 6)$ -triangle  $(U_1, V_1, W_1) = (544/105, 1995/136, 254659/14280)$  with area  $19\sqrt{3}$ . Also, the integer  $nm = 114$  is a  $2\pi/3$ -congruent number over  $\mathbb{Q}$  and  $(U_2, V_2, W_2) = (5, 912/10, 469/5)$  is a  $2\pi/3$ -triangle with area  $114\sqrt{3}$  from which we obtain a  $(\mathbb{Q}(\sqrt{6}), 2\pi/3, 19)$ -triangle

$$(5\sqrt{6}/6, 76\sqrt{6}/5, 469\sqrt{6}/30).$$

By a similar methods as in Example 4.2, one can find a  $(\mathbb{Q}(\sqrt{6}), 2\pi/3, 19)$ -triangle

$$\begin{aligned} (U, V, W)_{2\pi/3} = & \left( (25449816 + 4838521\sqrt{6})/4683550, \right. \\ & 20(4145193 - 12554399\sqrt{6})/28499829, \\ & \left. 7(3589965612532 - 2573211605723\sqrt{6})/1170880474675 \right), \end{aligned}$$

satisfying Theorem 1.4.

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